Chapter 8

Solutions to chapter 3 problems

Solution to Exercise 3.1.

a) We calculate the right-hand side of Eq. (3.5) using the decomposition (3.3) (we call the integration variable x'):

$$\langle x|\ \psi\rangle = \int_{-\infty}^{+\infty} \psi(x')\,\langle x|\ x'\rangle\,\mathrm{d}x' \overset{(3.1)}{=} \int_{-\infty}^{+\infty} \psi(x')\,\delta(x-x')\mathrm{d}x' = \psi(x). \tag{8.1}$$

The last equality follows from the definition of the delta function [see Eq. (C.3)].

b) Let us act with the operator $\hat{\mathbb{1}} = \int_{-\infty}^{+\infty} |x\rangle\langle x|\,\mathrm{d}x$ upon an arbitrary state $|\psi\rangle$. We have, according to the properties of the outer product,

$$\hat{I}\,|\psi\rangle = \int_{-\infty}^{+\infty} |x\rangle\,\langle x|\ \psi\rangle\,\mathrm{d}x \overset{(3.5)}{=} \int_{-\infty}^{+\infty} |x\rangle\,\psi(x)\mathrm{d}x \overset{(3.3)}{=} |\psi\rangle\,.$$

The operator $\hat{I}$ acting on any state returns the same state, i.e. it is the identity operator.

c) We insert the identity operator (3.6) into $\langle\psi_1|\ \psi_2\rangle$:

$$\begin{aligned}
\langle\psi_1|\ \psi_2\rangle &= \left\langle \psi_1 \middle| \left(\int_{-\infty}^{+\infty} |x\rangle\langle x|\,\mathrm{d}x\right) \middle|\ \psi_2\right\rangle \\
&= \int_{-\infty}^{+\infty} \langle\psi_1|\ x\rangle\,\langle x|\ \psi_2\rangle\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \psi_1^*(x)\psi_2(x)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} |\psi(x)|^2 dx.
\end{aligned}$$

Solution to Exercise 3.2. All the relations from Ex. 3.1 rely upon Eq. (3.5), which, in turn, is a consequence of Eq. (3.1). If the latter relation were replaced by $\langle x|\ x'\rangle = \epsilon(x-x') = \begin{cases} 1 \text{ if } x=x' \\ 0 \text{ if } x\neq x' \end{cases}$, Eq. (8.1) would take the form

$$\langle x|\ \psi\rangle = \int_{-\infty}^{+\infty} \psi(x')\,\langle x|\ x'\rangle\,\mathrm{d}x' = \int_{-\infty}^{+\infty} \psi(x')\,\epsilon(x-x')\mathrm{d}x' = 0.$$

The last equality is valid because the function $\psi(x')\epsilon(x-x')$ takes a nonzero value at only one point in $\mathbb{R}$ and its value its finite. The integral of such a function is zero.

Solution to Exercise 3.3. Applying Eq. (3.7), we find that

$$\langle\psi|\ \psi\rangle = \int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x.$$

The left-hand side of this equation is 1 because $|\psi\rangle$ is a physical state.

Solution to Exercise 3.4.

a) Integrating the squared absolute value of the wavefunction over the real axis we have

$$\int_{-\infty}^{+\infty} |\psi_{\mathrm{top-hat}}(x)|^2 \mathrm{d}x = A^2 \int_a^b \mathrm{d}x = A^2(b-a),$$

and thus

$$A = \frac{1}{\sqrt{b-a}}.$$

b) using Eq. (C.9) we find

$$\int_{-\infty}^{+\infty} |\psi_{\mathrm{Gauss}}(x)|^2 \mathrm{d}x = A^2 \int_{-\infty}^{+\infty} e^{-\frac{x^2}{d^2}} \mathrm{d}x = A^2 \int_{-\infty}^{+\infty} e^{-\frac{x^2}{d^2}} \mathrm{d}x = \sqrt{\pi} d,$$

so

$$A = \frac{1}{\pi^{1/4}\sqrt{d}}.$$

Solution to Exercise 3.5. According to Eq. (3.5), the wavefunction of the state $|x_0\rangle$ is

$$\langle x|\ x_0\rangle = \delta(x - x_0).$$

Solution to Exercise 3.6. In accordance with the definition of the continuous-variable observable,

$$\hat{x}\,|x\rangle = \left(\int_{-\infty}^{+\infty} x'\,|x'\rangle\,\langle x'|\,\mathrm{d}x'\right)|x\rangle = \int_{-\infty}^{+\infty} x'\,|x'\rangle\,\langle x'|\ x\rangle\,\mathrm{d}x' = \int_{-\infty}^{+\infty} x'\,|x'\rangle\,\delta(x'-x)\mathrm{d}x' = x\,|x\rangle\,.$$

Solution to Exercise 3.7.

a) We insert the identity operator (3.6) at both sides of $\hat{A}$:

$$\begin{aligned}
\hat{A} &= \hat{\mathbf{1}}\hat{A}\hat{\mathbf{1}} \\
&= \left(\int_{-\infty}^{+\infty} |x\rangle\langle x|\right)\mathrm{d}x)\hat{A}\left(\int_{-\infty}^{+\infty} |x'\rangle\langle x'|\right)\mathrm{d}x') \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x\rangle\left\langle x\right|\ \hat{A}\left|\ x'\right\rangle\langle x'|\,\mathrm{d}x\mathrm{d}x' \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(x,x')\,|x\rangle\langle x'|\,\mathrm{d}x\mathrm{d}x'.
\end{aligned}$$

b) Using the same approach,

$$\begin{aligned}
\langle\psi|\,\hat{A}\,|\psi\rangle &= \langle\psi|\left(\int_{-\infty}^{+\infty} |x\rangle\langle x|\,\mathrm{d}x\right)\hat{A}\left(\int_{-\infty}^{+\infty} |x'\rangle\langle x'|\,\mathrm{d}x'\right)|\psi\rangle \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle\psi|\ x\rangle\left\langle x\right|\ \hat{A}\left|\ x'\right\rangle\langle x'|\ \psi\rangle\ \mathrm{d}x\,\mathrm{d}x' \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi^*(x)A(x,x')\psi(x')\,\mathrm{d}x\,\mathrm{d}x'.
\end{aligned}$$

c) Here we insert the identity operator only at the right-hand side of $\hat{A}$:

$$\begin{aligned}\langle x|\,\hat{A}\,|\psi\rangle &= \langle x|\,\hat{A}\left(\int_{-\infty}^{+\infty}|x'\rangle\langle x'|\,\mathrm{d}x'\right)|\psi\rangle \\ &= \int_{-\infty}^{+\infty}\left\langle x\right|\hat{A}\left|x'\right\rangle\langle x'|\,\psi\rangle\,\mathrm{d}x' \\ &= \int_{-\infty}^{+\infty}A(x,x')\psi(x')\,\mathrm{d}x'.\end{aligned}$$

d) Similarly,

$$\begin{aligned}\langle\psi|\,\hat{A}\,|x\rangle &= \langle\psi|\left(\int_{-\infty}^{+\infty}|x'\rangle\langle x'|\,\mathrm{d}x'\right)\hat{A}\,|x\rangle \\ &= \int_{-\infty}^{+\infty}\langle\psi|\,x'\rangle\left\langle x'\right|\hat{A}\left|x\right\rangle\,\mathrm{d}x' \\ &= \psi^*(x')\int_{-\infty}^{+\infty}A(x,x')\,\mathrm{d}x'.\end{aligned}$$

e) According to the properties of adjoint operators (see Ex. 1.74),

$$(A^\dagger)(x,x') = \left\langle x\right|\hat{A}^\dagger\left|x'\right\rangle = \left\langle x'\right|\hat{A}\left|x\right\rangle^* = A^*(x',x).$$

f) Inserting the identity operator between $\hat{A}$ and $\hat{B}$, we find

$$\begin{aligned}\langle x|\,\hat{A}\hat{B}\,|x'\rangle &= \langle x|\,\hat{A}\left(\int_{-\infty}^{+\infty}|x''\rangle\langle x''|\,\mathrm{d}x''\right)\hat{B}\,|x'\rangle \\ &= \int_{-\infty}^{+\infty}\left\langle x\right|\hat{A}\left|x''\right\rangle\left\langle x''\right|\hat{B}\left|x'\right\rangle\,\mathrm{d}x'' \\ &= \int_{-\infty}^{+\infty}A(x,x'')B(x'',x')\mathrm{d}x''.\end{aligned}$$

Solution to Exercise 3.8. Using the definition (3.11) of the continuous-variable observable,

$$\begin{aligned}\langle\psi|\,\hat{x}\,|\psi\rangle &= \langle\psi|\left(\int_{-\infty}^{+\infty}x\,|x\rangle\langle x|\,\mathrm{d}x\right)|\psi\rangle \\ &= \int_{-\infty}^{+\infty}x\,\langle\psi|\,x\rangle\,\langle x|\,\psi\rangle\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty}x\,|\psi(x)|^2\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty}x\,\mathrm{pr}(x)\,\mathrm{d}x.\end{aligned}$$

Solution to Exercise 3.10.

a) If a 1000-kg car is moving at a velocity 20 m/s (72 km/h), its momentum is $p = 2\times10^4$ kg·m/s. Using the tabulated value of $2\pi\hbar = 6.6\times10^{-34}$ m$^2\cdot$ kg / s, we find the de Broglie wavelength equal to $\lambda = 2\pi\hbar/p = 3.3\times10^{-38}$ m.

b) The average translational velocity of molecules of a gas is $v = \sqrt{3k_BT/m}$, and the momentum is $p = \sqrt{3k_BTm}$, where k_B is the Boltzmann constant ($k = 1.38\times10^{-23}$ J/K, $T = 300$ K is the room temperature and $m = M_{\mathrm{air}}/N_A = 4.7\times10^{-26}$ kg is the average molecular mass (where $M_{\mathrm{air}} = 0.028$ kg/mol is the molar mass of air and $N_A = 6\times10^{23}$ is the Avogadro number). We find $p = 2.4\times10^{-23}$ kg·m/s and hence $\lambda = 2.7\times10^{-11}$ m.

c) The kinetic energy of the electron is $p^2/2m = eU$, where $m = 9.1 \times 10^{-31}$ kg is the electron mass, $e = 1.6 \times 10^{-19}$ Col is the electron charge and $U = 10^5$ V is the voltage. We find $p = 1.9 \times 10^{-22}$ kg·m/s and $\lambda = 3.5 \times 10^{-12}$ m. Because the de Broglie wavelength of the electron is much smaller than the wavelength of light, the electron microscope achieves much higher resolution than optical.

d) By analogy with part (b), we find the mass of rubidium atoms as $m = 0.087/(6 \times 10^{23})$ kg=1.5×10^{-25} kg and their momentum $p = \sqrt{3k_BTm} = 7.9 \times 10^{-28}$ kg·m/s. The de Broglie wavelength is 8.3×10^{-7} m=0.78 μm. This wavelength is comparable to the distance between atoms in the condensate, which leads to quantum effects in interaction between atoms.

Solution to Exercise 3.11. Using the method of "inserting identity", we write

$$|p\rangle = \int_{-\infty}^{+\infty} |x\rangle \langle x|\ p\rangle \,\mathrm{d}x \overset{(3.24)}{=} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{i\frac{px}{\hbar}} |x\rangle \,\mathrm{d}x.$$

Equation (3.28) is proven similarly.

Solution to Exercise 3.12. According to Eq. (3.7),

$$\langle p|\ p'\rangle = \int_{-\infty}^{+\infty} \psi^*_{|p\rangle}(x)\psi_{|p'\rangle}(x)\mathrm{d}x = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{i\frac{p'-p}{\hbar}x}\mathrm{d}x \overset{(C.19)}{=} \frac{1}{2\pi\hbar}2\pi\hbar\delta(p'-p) = \delta(p'-p)$$

.

Solution to Exercise 3.13. To convert between the position and momentum bases, we apply our usual trick of inserting the identity operator:

$$\begin{aligned}
\psi(x) &= \langle x|\ \psi\rangle = \langle x| \left(\int_{-\infty}^{+\infty} |p\rangle\langle p|\ \mathrm{d}p\right)|\psi\rangle \\
&= \int_{-\infty}^{+\infty} \langle x|\ p\rangle \langle p|\ \psi\rangle\ \mathrm{d}p \\
&\overset{(3.24)}{=} \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} e^{i\frac{px}{\hbar}}\psi(p)\,\mathrm{d}p;
\end{aligned}$$

$$\begin{aligned}
\psi(p) &= \langle p|\ \psi\rangle = \langle p| \left(\int_{-\infty}^{+\infty} |x\rangle\langle x|\ \mathrm{d}x\right)|\psi\rangle \\
&= \int_{-\infty}^{+\infty} \langle p|\ x\rangle \langle x|\ \psi\rangle\ \mathrm{d}x \\
&\overset{(3.24)}{=} \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} e^{-i\frac{px}{\hbar}}\psi(x)\,\mathrm{d}x.
\end{aligned}$$

Solution to Exercise 3.16.

a) Again, we use the wavevector basis rather than the momentum basis:

$$\begin{aligned}
\tilde{\psi}(k) &= \frac{1}{(\pi d^2)^{1/4}}\mathcal{F}[e^{i\frac{p_0x}{\hbar}}e^{-\frac{(x-a)^2}{2d^2}}](k) \\
&\overset{(C.25)}{=} \frac{1}{(\pi d^2)^{1/4}}\mathcal{F}[e^{-\frac{(x-a)^2}{2d^2}}](k-k_0) \qquad (\text{where } k_0 = \frac{p_0}{\hbar}) \\
&\overset{(C.24)}{=} \frac{1}{(\pi d^2)^{1/4}}e^{-i(k-k_0)a}\mathcal{F}[e^{-\frac{x^2}{2d^2}}](k-k_0) \\
&\overset{(C.15)}{=} \frac{1}{(\pi d^2)^{1/4}}e^{-i(k-k_0)a}de^{-(k-k_0)^2d^2/2} \\
&= \frac{\sqrt{d}}{\pi^{1/4}}e^{-i(k-k_0)a}e^{-(k-k_0)^2d^2/2}.
\end{aligned}$$

Now we can rewrite this result in the momentum basis using Eq. (3.36):

$$\tilde{\psi}(p) = \frac{\sqrt{d}}{\pi^{1/4}\sqrt{\hbar}} e^{-i(p-p_0)a/\hbar} e^{-(p-p_0)^2 d^2/2\hbar^2}. \tag{8.2}$$

b) The probability density

$$\mathrm{pr}(x) = |\psi(x)|^2 = \frac{1}{\sqrt{\pi}d} e^{-\frac{(x-a)^2}{d^2}}$$

is a Gaussian curve centered around $x = a$. The expectation of the position (i.e. the *average* value of the position observed) must be at $\langle x \rangle = a$. Similarly, $\langle k \rangle = k_0$ and thus $\langle p \rangle = p_0$.

Solution to Exercise 3.17. Writing the momentum observable as $\hat{p} = \int_{-\infty}^{+\infty} p\, |p\rangle\langle p|\, \mathrm{d}p$, we find

$$\langle x|\, \hat{p}\, |\, x' \rangle = \int_{-\infty}^{+\infty} p\, \langle x|\, p \rangle\, \langle p|\, x' \rangle\, \mathrm{d}p = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} p\, e^{i\frac{p}{\hbar}(x-x')} \mathrm{d}p.$$

We now notice that the integrand can be expressed as $p\, e^{-i\frac{p}{\hbar}(x-x')} = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} e^{i\frac{p}{\hbar}(x-x')}$. The order of differentiation and integration can be inverted:

$$\begin{aligned}
\langle x|\, \hat{p}\, |\, x' \rangle &= \frac{1}{2\pi\hbar}(-i)\hbar \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{+\infty} e^{i\frac{p}{\hbar}(x-x')} \mathrm{d}p \\
&\overset{(C.19)}{=} \frac{1}{2\pi\hbar}(-i)\hbar \frac{\mathrm{d}}{\mathrm{d}x} (2\pi\hbar)\delta(x-x') \\
&= -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \delta(x-x').
\end{aligned}$$

Solution to Exercise 3.18. Using the result of the Ex. 3.17,

$$\begin{aligned}
\langle x|\, \hat{p}\, |\, \psi \rangle &= \langle x|\, \hat{p} \left(\int_{-\infty}^{+\infty} |x'\rangle\langle x'|\, \mathrm{d}x' \right) |\psi\rangle \\
&= \int_{-\infty}^{+\infty} \langle x|\, \hat{p}\, |\, x' \rangle\, \langle x'|\, \psi \rangle\, \mathrm{d}x' \\
&= -i\hbar \int_{-\infty}^{+\infty} \left[\frac{\mathrm{d}}{\mathrm{d}x} \delta(x-x') \right] \psi(x')\, \mathrm{d}x' \\
&= -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \left[\int_{-\infty}^{+\infty} \delta(x-x')\psi(x')\, \mathrm{d}x' \right] \\
&= -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \psi(x)
\end{aligned}$$

Solution to Exercise 3.19. Using the result of Ex. 3.17 and 3.18,

$$\begin{aligned}
\langle x|\, \hat{p}^2\, |\, \psi \rangle &= \langle x|\, \hat{p} \left(\int_{-\infty}^{+\infty} |x'\rangle\langle x'|\, \mathrm{d}x' \right) \hat{p}\, |\psi\rangle \\
&= \int_{-\infty}^{+\infty} \langle x|\, \hat{p}\, |\, x' \rangle\, \langle x'|\, \hat{p}\, |\, \psi \rangle\, \mathrm{d}x' \\
&= -i\hbar \int_{-\infty}^{+\infty} \left[\frac{\mathrm{d}}{\mathrm{d}x} \delta(x-x') \right] \left[-i\hbar \frac{\mathrm{d}}{\mathrm{d}x'} \psi(x') \right] \mathrm{d}x' \\
&= (-i\hbar)^2 \frac{\mathrm{d}}{\mathrm{d}x} \left[\int_{-\infty}^{+\infty} \delta(x-x') \frac{\mathrm{d}}{\mathrm{d}x'} \psi(x')\, \mathrm{d}x' \right] \\
&= -\hbar^2 \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\mathrm{d}}{\mathrm{d}x} \psi(x) \right] \\
&= -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \psi(x)
\end{aligned}$$

Solution to Exercise 3.20. Left for the reader as an independent exercise.

Solution to Exercise 3.21.

a)

$$\begin{aligned} e^{-i\hat{p}x_0/\hbar}\,|x\rangle &\overset{(3.27)}{=} \frac{1}{2\pi\hbar}e^{-i\hat{p}x_0/\hbar}\int_{-\infty}^{+\infty} e^{-i\hat{p}x/\hbar}\,|p\rangle\,\mathrm{d}p \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{-i\hat{p}(x+x_0)/\hbar}\,|p\rangle\,\mathrm{d}p \\ &= |x+x_0\rangle\,. \end{aligned}$$

b) since the operator displaces each position eigenstate by x_0, it displaces the whole wavefunction $\psi(x)$ as shown in Fig. 3.2. The new wavefunction is then given by

$$\begin{aligned} \psi'(x) &= \langle x|\, e^{-i\hat{p}x_0/\hbar}\,|\psi\rangle \\ &= \int_{-\infty}^{+\infty} \langle x|\, e^{-i\hat{p}x_0/\hbar}\,|x'\rangle\,\langle x'|\,\psi\rangle\,\mathrm{d}x' \\ &\overset{(3.44)}{=} \int_{-\infty}^{+\infty} \langle x|\, x'+x_0\rangle\,\langle x'|\,\psi\rangle \\ &= \int_{-\infty}^{+\infty} \delta(x'+x_0-x)\psi(x')\mathrm{d}x' \\ &= \psi(x-x_0). \end{aligned}$$

c) Since the operator displaces the whole wavefunction by x_0, it must also adds x_0 to the mean position value. Formally this can be expressed as follows. For the mean position value in the state $|\psi'\rangle = e^{-i\hat{p}x_0/\hbar}\,|\psi\rangle$, we have

$$\begin{aligned} \langle x\rangle_{|\psi'\rangle} &= \int_{-\infty}^{+\infty} x|\psi(x-x_0)|^2\mathrm{d}x \\ &\overset{x'=x-x_0}{=} \int_{-\infty}^{+\infty} x'|\psi(x')|^2\mathrm{d}x + x_0\int_{-\infty}^{+\infty}|\psi(x')|^2\mathrm{d}x' \\ &= \langle x\rangle_{|\psi\rangle} + x_0, \end{aligned} \tag{8.3}$$

where $\langle x\rangle_{|\psi\rangle}$ is the mean position value in state $|\psi\rangle$.

To find the transformation of the mean momentum value under the displacement operator, we notice that the action of this operator in the momentum basis is simply a multiplication

$$\tilde{\psi}(p) \overset{e^{-i\hat{p}x_0/\hbar}}{\longmapsto} \tilde{\psi}'(p) = e^{-ipx_0/\hbar}\tilde{\psi}(p). \tag{8.4}$$

Hence $|\tilde{\psi}'(p)|^2 = |\tilde{\psi}(p)|^2$ and

$$\begin{aligned} \langle p\rangle_{|\psi'\rangle} &= \int_{-\infty}^{+\infty} p|\tilde{\psi}'(p)|^2\mathrm{d}p \\ &= \int_{-\infty}^{+\infty} p|\tilde{\psi}(p)|^2\mathrm{d}p \\ &= \langle p\rangle_{|\psi\rangle}\,. \end{aligned} \tag{8.5}$$

d) The fact that the uncertainties of the position and momentum of the displaced state are the same as those in the original state is, again, intuitive (Fig. 3.2). A rigorous proof can be dome

as follows. For the position:

$$\begin{aligned}
\left\langle \Delta x^2 \right\rangle_{|\psi'\rangle} &= \left\langle \left(x - \langle x \rangle_{|\psi'\rangle} \right)^2 \right\rangle_{|\psi'\rangle} \\
&= \int_{-\infty}^{+\infty} \left(x - \langle x \rangle_{|\psi\rangle} - x_0 \right)^2 |\psi(x - x_0)|^2 \mathrm{d}x \\
&\overset{x'=x-x_0}{=} \int_{-\infty}^{+\infty} \left(x' - \langle x \rangle_{|\psi\rangle} \right)^2 |\psi(x')|^2 \mathrm{d}x' \\
&= \left\langle \left(x - \langle x \rangle_{|\psi\rangle} \right)^2 \right\rangle_{|\psi\rangle} \\
&= \left\langle \Delta x^2 \right\rangle_{|\psi\rangle} . \qquad (8.6)
\end{aligned}$$

For the momentum:

$$\begin{aligned}
\left\langle \Delta p^2 \right\rangle_{|\psi'\rangle} &= \left\langle \left(p - \langle p \rangle_{|\psi'\rangle} \right)^2 \right\rangle_{|\psi'\rangle} \\
&= \int_{-\infty}^{+\infty} \left(p - \langle p \rangle_{|\psi\rangle} \right)^2 |\tilde{\psi}'(p)|^2 \mathrm{d}p \\
&= \int_{-\infty}^{+\infty} \left(p - \langle p \rangle_{|\psi\rangle} \right)^2 |\tilde{\psi}(p)|^2 \mathrm{d}p \\
&= \left\langle \Delta p^2 \right\rangle_{|\psi\rangle} \qquad (8.7)
\end{aligned}$$

Solution to Exercise 3.22.

$$\begin{aligned}
e^{i\hat{x}p_0/\hbar} |p\rangle &= \frac{1}{2\pi\hbar} e^{i\hat{x}p_0/\hbar} \int_{-\infty}^{+\infty} e^{i\hat{p}x/\hbar} |x\rangle \, \mathrm{d}x \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\hat{x}(p+p_0)/\hbar} |x\rangle \, \mathrm{d}x \\
&= |p + p_0\rangle .
\end{aligned}$$

Other properties are proven similarly to Ex. 3.21.

Solution to Exercise 3.24. We recall that the probability to detect a certain value of momentum is

$$\mathrm{pr}(p) = \langle p | \, \psi \rangle = |\tilde{\psi}(p)|^2,$$

where the wavefunction $\tilde{\psi}(p)$ in the momentum basis is the Fourier transform of the wavefunction $\psi(x)$ in the position basis. Because the latter is real, $\tilde{\psi}(p) = \tilde{\psi}(-p)$ (Ex. C.6) and thus $\mathrm{pr}(p) = \mathrm{pr}(-p)$.

The expectation value of the momentum observable is given by

$$\langle p \rangle = \int_{-\infty}^{+\infty} p \, \mathrm{pr}(p) \mathrm{d}p = 0$$

because $p \, \mathrm{pr}(p)$ is an odd function.

Solution to Exercise 3.25.

a) Because the position operator is Hermitian, $\langle x | \, \hat{x} = x \, \langle x |$ and thus

$$\langle x | \, \hat{x}\hat{p} | \, \psi \rangle = \hat{x} \, \langle x | \, \hat{p} | \, \psi \rangle \overset{(3.18)}{=} -i\hbar x \frac{\mathrm{d}}{\mathrm{d}x} \psi(x).$$

b) Let us denote $\hat{x}\,|\psi\rangle = |\phi\rangle$; then $\phi(x) = \langle x|\,\hat{x}|\,\psi\rangle = x\psi(x)$. Therefore

$$\langle x|\,\hat{p}\hat{x}|\,\psi\rangle = \langle x|\,\hat{p}|\,\phi\rangle = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}[x\psi(x)] = -i\hbar\psi(x) - i\hbar x\frac{\mathrm{d}}{\mathrm{d}x}\psi(x).$$

Note that the above relation can also be found using the technique of "inserting identity". The reader can try this independently.

c) Using the two results above, we find

$$\langle x|\,[\hat{x},\hat{p}]|\,\psi\rangle = \langle x|\,\hat{x}\hat{p}|\,\psi\rangle - \langle x|\,\hat{p}\hat{x}|\,\psi\rangle = i\hbar\psi(x).$$

Therefore, applying the operator $[\hat{x},\hat{p}]$ to any $|\psi\rangle$ is equivalent to multiplying this state by $i\hbar$. We conclude that $[\hat{x},\hat{p}] = i\hbar\hat{\mathbf{1}}$.

According to the uncertainty principle, a nonzero commutator between the position and momentum means that these two quantities can never be determined precisely simultaneously:

$$\left\langle \Delta\hat{x}^2\right\rangle \left\langle \Delta\hat{p}^2\right\rangle \geq \frac{1}{4}\hbar^2. \tag{8.8}$$

Solution to Exercise 3.26. Substituting the uncertainty principle (1.61), we find:

$$\left\langle \Delta\hat{x}^2\right\rangle \left\langle \Delta\hat{p}^2\right\rangle \geq \frac{1}{4}\left|\langle i\hbar\hat{\mathbf{1}}\rangle\right|^2 = \frac{1}{4}\hbar^2. \tag{8.9}$$

Solution to Exercise 3.27. To find Δx^2 we use the definition (1.53) of the mean square uncertainty, as well as the fact that $\langle x\rangle = 0$ in accordance with Ex. 3.16:

$$\begin{aligned}
\langle \Delta x^2\rangle &= \int_{-\infty}^{+\infty} x^2 \mathrm{pr}(x)\mathrm{d}x \\
&= \frac{1}{\sqrt{\pi}d}\int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{d^2}}\mathrm{d}x \\
&= \frac{1}{\sqrt{\pi}d}\int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{d^2}}\mathrm{d}x \qquad \text{(replace variable: } t = x/d\text{)} \\
&= \frac{d^3}{\sqrt{\pi}d}\int_{-\infty}^{+\infty} t^2 e^{-t^2}\mathrm{d}t \\
&\overset{(3.50)}{=} \frac{d^2}{2}.
\end{aligned}$$

Similarly, Eq. (3.39), $\langle \Delta k^2\rangle = 1/2d^2$ and thus $\langle \Delta p^2\rangle = \hbar^2/2d^2$. The product of the uncertainties is

$$\langle \Delta x^2\rangle\langle \Delta p^2\rangle = \frac{\hbar^2}{4},$$

which is the minimum allowed by the uncertainty principle.

Solution to Exercise 3.28.

a) The wavefucntion in the momentum representation (for convenience, we use the physically identical wavevector representation) can be found using the standard conversion formula (3.35).

The Fourier transformation has to be applied to both x_A and x_B.

$$\begin{aligned}
\tilde{\Psi}(k_A, k_b) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi(x_A, x_B)e^{-ik_A x_A}e^{-ik_B x_B}\mathrm{d}x_A\mathrm{d}x_B \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(x_A - x_B)e^{-ik_A x_A - ik_B x_B}\mathrm{d}x_A\mathrm{d}x_B \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ik_A x_A - ik_B x_A}\mathrm{d}x_A \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-i(k_A+k_B)x_A}\mathrm{d}x_A \\
&\overset{\text{(C.18)}}{=} \frac{1}{2\pi}2\pi\delta(k_A + k_B) \\
&= \delta(k_A + k_B).
\end{aligned}$$

b) The wavefunction $\Psi(x_A, x_B) = \delta(x_A - x_B)$ of the system in the position basis implies that the positions of Alice's and Bob's particles must be identical. If Alice detects her particle at a position x_0, Bob's particle will project onto a state with the same position, i.e. $|x_0\rangle$.

c) Similarly, because $\tilde{\Psi}(k_A, k_b) = \delta(k_A + k_B)$, Alice's detection of wavevector k_0 (or momentum $p_0 = \hbar k_0$) will project Bob's state onto $|-k_0\rangle$ (or $|-p_0\rangle$).

Solution to Exercise 3.29. We rewrite the Schrödinger equation

$$|\dot{\psi}\rangle = -\frac{i}{\hbar}\hat{H}\,|\psi\rangle = -\frac{i}{\hbar}\left[V(\hat{x}) + \frac{\hat{p}^2}{2m}\right]|\psi\rangle$$

in the position basis using the result of Ex. 3.19:

$$\begin{aligned}
\dot{\psi}(x,t) &= \left\langle x\right|\left.\dot{\psi}\right\rangle \\
&= -\frac{i}{\hbar}\left\langle x\right|V(\hat{x})\left|\psi\right\rangle - \frac{i}{\hbar}\left\langle x\right|\left.\frac{\hat{p}^2}{2m}\right|\psi\right\rangle \\
&= -\frac{i}{\hbar}V(x)\psi(x,t) - \frac{i}{\hbar}\frac{-\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\psi(x,t).
\end{aligned}$$

Solution to Exercise 3.30. In the absence of potential, the Hamiltonian is a function of the momentum: $\hat{H} = \hat{p}^2/2m$. An eigenstate $|p\rangle$ of the momentum is therefore automatically an energy eigenstate with the eigenvalue $E = p^2/2m$. According to the general solution (1.77) of the Schrödinger equation, this state evolves as follows:

$$|p\rangle \to e^{-\frac{i}{\hbar}Et}\,|p\rangle = e^{-i\frac{p^2}{2m\hbar}t}\,|p\rangle\,.$$

Assuming that the wavefunction of the momentum eigenstate at the moment $t = 0$ is given by the de Broglie wave (3.24), its evolution can be written in the position basis as

$$\psi_{|p\rangle}(x,t) = \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{px}{\hbar} - i\frac{p^2}{2m\hbar}t}.$$

To find the phase velocity, we rewrite the above as

$$\psi_{|p\rangle}(x,t) = \frac{1}{\sqrt{2\pi\hbar}}e^{ikx - i\omega t} = \frac{1}{\sqrt{2\pi\hbar}}e^{ik\left(x - \frac{\omega}{k}t\right)},$$

where $k = p/\hbar$ is the wavevector and $\omega = p^2/2m\hbar$ is the oscillation frequency. Over the time t, the wave translates by a distance $x = (\omega/k)t = (p/2m)t$, so the phase velocity is $v_{ph} = \omega/k = p/2m$.

Solution to Exercise 3.31.

a) We found the wavefunction in the wavevector representation in Ex. 3.16:

$$\tilde{\psi}(k) = \frac{\sqrt{d}}{\pi^{1/4}} e^{-(k-k_0)^2 d^2/2}.$$

Since $|\psi\rangle = \int_{-\infty}^{+\infty} \psi(k)\,|k\rangle\,\mathrm{d}k$ and each wavevector eigenstate $|k\rangle$ is also an eigenstate of the Hamiltonian with eigenvalue $E_k = \hbar^2 k^2/2m$, we have for the evolution of state $|\psi\rangle$

$$\tilde{\psi}(k,t) = \frac{\sqrt{d}}{\pi^{1/4}} e^{-(k-k_0)^2 d^2/2} e^{-i\hbar k^2 t/2m}. \tag{8.10}$$

b) Defining $k = k_0 + \delta k$, we rewrite Eq. (8.10) as

$$\tilde{\psi}(k,t) = \frac{\sqrt{d}}{\pi^{1/4}} e^{-i\frac{\hbar k_0^2 t}{2m}} e^{-i\frac{\hbar k_0 \delta k t}{m}} e^{-\delta k^2\left(\frac{d^2}{2} - i\frac{\hbar t}{2m}\right)}. \tag{8.11}$$

Let us now inverse Fourier transform this result back into the position basis. The first exponential in the equation above is a k-independent phase factor, which is not affected by the Fourier transform. The last exponential is a Gaussian function, whose (inverse) Fourier transform is also a Gaussian. The second exponential is a linear phase factor, whose effect on the Fourier transform translates, according to Eq. (C.25), into the shift of the position by $\hbar k_0 t/m$. Finally, since the whole expression Eq. (8.11) is given in terms of $\delta k = k - k_0$, we must apply (C.24), obtaining the factor $e^{ik_0 x}$. The resulting wavefunction is as follows:

$$\psi(x,t) = \frac{\sqrt{d}}{\pi^{1/4}} \left(d^2 - i\frac{\hbar t}{m}\right)^{-1/2} e^{-i\frac{\hbar k_0^2 t}{2m}} e^{ik_0 x} e^{-\left(x - \frac{\hbar k_0 \delta k t}{m}\right)^2 \Big/ 2\left(d^2 - i\frac{\hbar t}{m}\right)}, \tag{8.12}$$

c) Expression (8.12) represents a Gaussian wavepacket centered at $\langle x\rangle = (p_0/m)t$ according to Ex. 3.16. The finding of the position uncertainty proceeds similarly to Ex. 3.27, but we have to take into account the complexity of the Gaussian exponent in Eq. (8.12). We find

$$\mathrm{pr}_x = \psi^*(x,t)\psi(x,t) = \frac{d}{\sqrt{\pi}} \left(d^4 + \frac{\hbar^2 t^2}{m^2}\right)^{-1/2} e^{-d^2\left(x - \frac{\hbar k_0 \delta k t}{m}\right)^2 \Big/ 2\left(d^4 + \frac{\hbar^2 t^2}{m^2}\right)} \tag{8.13}$$

Using the integral (3.50) we find

$$\langle \Delta x^2 \rangle = \int_{-\infty}^{+\infty} (x - \langle x\rangle) p r_x \mathrm{d}x = \frac{d^2}{2}\left(1 + \frac{\hbar^2 t^2}{m^2 d^4}\right). \tag{8.14}$$

Solution to Exercise 3.32.

a) According to Eq. (8.14), the width of the Gaussian wavepacket behaves for large t according to

$$\sqrt{\langle \Delta x^2\rangle} \sim \frac{\hbar}{md} t. \tag{8.15}$$

The condition that p_0 greatly exceeds the momentum uncertainty of the initial wavepacket means, in accordance with Ex. 3.27, that $p_0 \gg \hbar/d$. This means that the traveled distance, $p_0 t/m$, is much greater than $\hbar t/md$, i.e. it is much greater than $\sqrt{\langle \Delta x^2\rangle}$ in accordance with Eq. (8.15).

b) We rewrite Eq. (8.15) as $t \sim \sqrt{\langle \Delta x^2\rangle} md/\hbar$. Substituting $\sqrt{\langle \Delta x^2\rangle} = 10^{-3}$ m, $d = 10^{-10}$ m and $m \approx 10^{-30}$ kg we find $t \approx 1$ ns.

Solution to Exercise 3.33. In accordance with Eq. (3.55), we have

$$v_{\rm gr} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\hbar k}{m}.$$

For all de Broglie waves constituting a wavepacket with a mean momentum p_0 and a small momentum uncertainty, we have $k \approx p_0/\hbar$ and hence $v_{\rm gr} = p_0/m$. This result is identical to the classical one.

Solution to Exercise 3.34. The time derivative of the operator's expectation value can be found using the rule for the derivative of a product:

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle A\rangle = \frac{\mathrm{d}}{\mathrm{d}x}\left\langle \psi\right|\hat{A}\left|\psi\right\rangle = \left\langle \dot{\psi}\right|\hat{A}\left|\psi\right\rangle + \left\langle \psi\right|A\left|\dot{\psi}\right\rangle. \tag{8.16}$$

The time derivative of the state $|\psi\rangle$ is determined from the Schrödinger equation: $|\dot{\psi}\rangle = (-i/\hbar)\hat{H}\,|\psi\rangle$. Because the Hamiltonian is a Hermitian operator, we also have $\langle\dot{\psi}| = (i/\hbar)\,\langle\psi|\,\hat{H}$. Substituting these relations into Eq. (8.16), we find

$$\langle\dot{A}\rangle = \frac{i}{\hbar}\left\langle \psi\right|\hat{H}\hat{A}\left|\psi\right\rangle - \frac{i}{\hbar}\left\langle \psi\right|\hat{A}\hat{H}\left|\psi\right\rangle = \frac{i}{\hbar}\left\langle \psi\right|[\hat{H},\hat{A}]\left|\psi\right\rangle.$$

Solution to Exercise 3.35.

a) According to the previous result, $\langle x\rangle = (i/\hbar)\langle[\hat{H},\hat{x}]\rangle$, so we need to find the commutator between the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ and the position operator $\hat{x}$. The potential energy is a function of the position and thus commutes with it. To find the commutator between the kinetic energy and the position, we use Eq. (1.52):

$$\left[\frac{\hat{p}^2}{2m},\hat{x}\right] = \frac{1}{2m}\big(\hat{p}[\hat{p},\hat{x}] + [\hat{p},\hat{x}]\hat{p}\big) \overset{\text{Ex. 3.48}}{=} -i\frac{\hat{p}\hbar}{m}$$

and thus $\langle x\rangle = \langle p\rangle/m$

b) Here we need to find the commutator between the Hamiltonian and the momentum operator; the only non-commuting term in the Hamiltonian is the potential energy $V(\hat{x})$. We accomplish our task by analyzing the action of the commutator $[V(\hat{x}),\hat{p}]$ on an arbitrary state $|\psi\rangle$ in the position basis. We set $|\phi\rangle = V(\hat{x})\,|\psi\rangle$ and write

$$\begin{aligned}
\langle x|\,[V(\hat{x}),\hat{p}]|\,\psi\rangle &= \langle x|\,V(\hat{x})\hat{p}|\,\psi\rangle - \langle x|\,\hat{p}V(\hat{x})|\,\psi\rangle \\
&= V(x)\,\langle x|\,\hat{p}|\,\psi\rangle - \langle x|\,\hat{p}|\,\phi\rangle \qquad \text{[because } |x\rangle \text{ is an eigenstate of } V(\hat{x})\text{]} \\
&= V(x)(-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) - (-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}[V(x)\psi(x)] \\
&= V(x)(-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) + i\hbar\frac{\mathrm{d}}{\mathrm{d}x}[V(x)]\psi(x) + i\hbar V(x)\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) \\
&= i\hbar\psi(x)\frac{\mathrm{d}}{\mathrm{d}x}V(x).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\langle\dot{p}\rangle &= \frac{i}{\hbar}\,\langle\psi|\,[V(\hat{x}),\hat{p}]|\,\psi\rangle \\
&= \frac{i}{\hbar}\int_{-\infty}^{+\infty}\langle\psi|\,x\rangle\,\langle x|\,[V(\hat{x}),\hat{p}]|\,\psi\rangle \\
&= -\int_{-\infty}^{+\infty}\psi^*(x)\psi(x)\frac{\mathrm{d}}{\mathrm{d}x}V(x)\mathrm{d}x.
\end{aligned}$$

The latter expression can be formally written as

$$\langle\dot{p}\rangle = -\left\langle \psi\right|\frac{d\hat{V}}{dx}\left|\psi\right\rangle.$$

Solution to Exercise 3.42. We can rewrite the time-independent Schrödinger equation (3.64) as

$$\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = (V_0 - E)\psi(x),$$

which can be simplified to

$$\frac{d^2}{dx^2}\psi(x) = \kappa^2\psi(x),$$

where $\kappa = \sqrt{2m(V_0 - E)}/\hbar$ does not depend on x. This second-order differential equation has two linearly-independent solutions:

$$\psi(x) = Ae^{\pm\kappa x}. \tag{8.17}$$

Note that κ is real only if $E < V_0$, i.e. the total energy is below the potential energy level. Otherwise, κ becomes imaginary and the solution (8.17) takes the form of the de Broglie wave:

$$\psi(x) = Ae^{\pm ikx}, \tag{8.18}$$

where $k = i\kappa = \sqrt{2m(E - V_0)}/\hbar$ is a real wavevector.

Solution to Exercise 3.43. Let us rewrite the time-independent Schrödinger equation (3.64) as follows:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = [E - V(x)]\psi(x).$$

If both $V(x)$ and $\psi(x)$ are finite for all x, so is the right-hand side of the above equation. This means that $\mathrm{d}^2\psi(x)/\mathrm{d}x^2$ is finite for all x as well. This implies in turn that the first derivative of the wavefunction is continuous for all x, because otherwise $\mathrm{d}^2\psi(x)/\mathrm{d}x^2$ would be singular at some points. Because $\mathrm{d}\psi(x)/\mathrm{d}x$ is continuous, it must be finite for all x. Therefore, $\psi(x)$ must be finite for all x as well.

Solution to Exercise 3.44.

a) Because the Hamiltonian $\hat{H}$ is a Hermitian operator, its eigenstates form an orthonormal basis. Our goal is to prove that such a basis can be composed of states with real wavefunctions.

Suppose this is not the case, i.e. there exists a Hamiltonian eigenstate $|\psi\rangle$ with eigenvalue E which cannot be expressed as a linear combination of eigenstates with real wavefunctions. Consider the state $|\psi^*\rangle$ described by the wavefunction $\psi^*(x)$, where $\psi(x)$ is the wavefunction of the state $|\psi\rangle$. If $\psi(x)$ is an eigenwavefunction of the Hamiltonian (i.e. satisfies the time-independent Schrödinger equation), so must $\psi^*(x)$; therefore, $|\psi^*\rangle$ must also be an eigenstate of the Hamiltonian with the same eigenvalue. Now consider the following states:

$$\begin{aligned} |\psi_1\rangle &= |\psi\rangle + |\psi^*\rangle\,; \\ |\psi_2\rangle &= [|\psi\rangle - |\psi^*\rangle]/i. \end{aligned}$$

According to properties of complex numbers, the wavefunctions of these states are real. Because $|\psi_1\rangle$ and $|\psi_2\rangle$ are linear combinations of $|\psi\rangle$ and $|\psi^*\rangle$, they are also eigenstates of $\hat{H}$ with eigenvalue E. Furthermore, the state $|\psi\rangle$ can be expressed as a linear combination

$$|\psi\rangle = \frac{1}{2}(|\psi_1\rangle + i\,|\psi_2\rangle)$$

of energy eigenstates with real eigenvalues. We have arrived at a contradiction.

b) By the same logic, consider an energy eigenstate $|\psi\rangle$ with eigenvalue E and wavefunction $\psi(x)$. If $\psi(x)$ satisfies the time-independent Schrödinger equation, the wavefunction $\psi(-x)$ does, too, so the state $|\psi^-\rangle$ with this wavefunction is also an eigenstate of the Hamiltonian. We now construct an energy eigenstate with an even wavefunction,

$$|\psi_1\rangle = |\psi\rangle + \left|\psi^-\right\rangle,$$

and one with an odd wavefunction,

$$|\psi_2\rangle = |\psi\rangle - \left|\psi^-\right\rangle .$$

The state $|\psi\rangle$ can then be expressed as a linear combination of $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$|\psi\rangle = \frac{1}{2}(|\psi_1\rangle + |\psi_2\rangle).$$

Solution to Exercise 3.45. As discussed in Ex. 3.42, energies E below a constant potential level V_0 are associated with eigenwavefunctions $\psi(x) = Ae^{\pm\kappa x}$, with $\kappa = \sqrt{2m(V_0 - E)}/\hbar$. Because of the normalization condition, the wavefunctions cannot have components that exponentially grow at infinity, and thus we must have

$$\psi(x) \to \begin{cases} Ae^{-\kappa x} \text{ at } x \to +\infty \\ A'e^{\kappa x} \text{ at } x \to -\infty \end{cases}$$

In other words, $\psi(x) \to 0$ for $|x| \to \pm\infty$, so we have a bound state.

Conversely, if the energy exceeds the potential at infinity, the eigenwavefunctions tend to $\psi(x) \to Ae^{ikx} + A'e^{-ikx}$, with $k = \sqrt{2m(E - V_0)}/\hbar$. If at least one of the factors A or A' does not vanish, the state is not bound.

Solution to Exercise 3.46. Because the potential is an even function of x, it suffices to look for even and odd solutions of the time-independent Schrödinger equation. Let us consider these two cases separately.

A general odd solution is of the form

$$\psi(x) = \begin{cases} -Be^{\kappa x}, & x < -a/2 \\ A\sin kx, & -a/2 \le x \le a/2 \\ Be^{-\kappa x}, & x > a/2 \end{cases} \tag{8.19}$$

with

$$k = \frac{\sqrt{2mE}}{\hbar}, \tag{8.20}$$

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \tag{8.21}$$

Because the potential is finite, both the wavefunction $\psi(x)$ and its derivative $\psi'(x)$ must be continuous. Writing these conditions for the boundary of the box $x = a/2$, we find[1]

$$\begin{aligned} A\sin kx\big|_{x=a/2} &= Be^{-\kappa x}|_{x=a/2}; \\ Ak\cos kx\big|_{x=a/2} &= -\kappa Be^{-\kappa x}|_{x=a/2} \end{aligned}$$

or

$$A\sin\frac{ka}{2} = Be^{-\kappa a/2}; \tag{8.22}$$

$$Ak\cos\frac{ka}{2} = -\kappa Be^{-\kappa a/2}. \tag{8.23}$$

These equations result in restrictions on energy values at which the time-independent Schrödinger equation has a solution. To see this, let us divide Eqs. (8.22) and (8.23) by each other. We obtain

$$\cot\frac{ka}{2} = -\frac{\kappa}{k}. \tag{8.24}$$

[1] The continuity condition for $x = -a/2$ yields the same set of equations.

This equation relates k and κ. Another relation between these quantities is due to Eq. (8.20), which can be incorporated into our calculations as follows. Let us denote $ka/2 = \theta$ and $\kappa a/2 = \theta_1$. From Eq. (8.20) we then have

$$\theta^2 + \theta_1^2 = \theta_0^2,$$

where

$$\theta_0 = \frac{\sqrt{2mV_0}}{\hbar}\frac{a}{2}. \tag{8.25}$$

Equation (8.24) now takes the form

$$\cot\theta = -\frac{\theta_1}{\theta} \tag{8.26}$$

or

$$-\cot\theta = \sqrt{\frac{\theta_0^2}{\theta^2} - 1}. \tag{8.27}$$

This equation contains only one unknown variable, θ, which is related to the evergy eigenvalue. Unfortunately, this equation is transcendental and cannot be solved in elementary functions.

A generic even solution is given by

$$\psi(x) = \begin{cases} Be^{\kappa x}, & x < -a/2 \\ A\cos kx, & -a/2 \le x \le a/2 \\ Be^{-\kappa x}, & x > a/2 \end{cases} \tag{8.28}$$

Proceeding in a fashion similar to the odd case, we find the continuity conditions for the boundary of the well

$$A\cos\frac{ka}{2} = Be^{-\kappa a/2}; \tag{8.29}$$

$$-Ak\sin\frac{ka}{2} = -\kappa Be^{-\kappa a/2}, \tag{8.30}$$

and the transcendental equation for θ

$$\tan\theta = \sqrt{\frac{\theta_0^2}{\theta^2} - 1}. \tag{8.31}$$

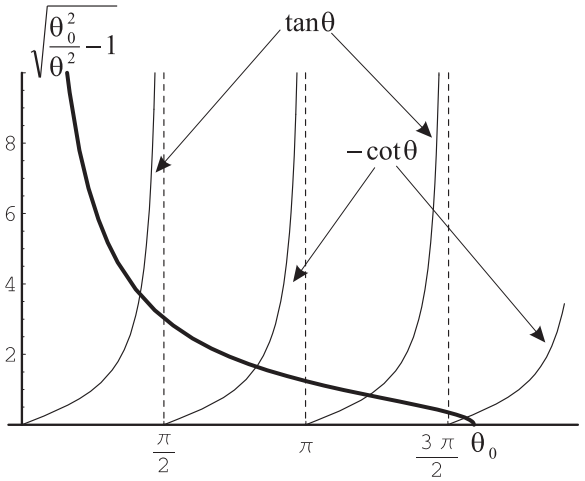


Figure 8.1: Graphic solution to transcendental equations (8.27) and (8.31) for $\theta_0 = 5$.

Solution to Exercise 3.47. When V_0 is infinite, so is the right-hand side of Eqs. (8.27) and (8.31). The tangent in the left-hand side of Eq. (8.31) takes on a positive infinite value when

$\theta = (2j+1)\pi/2$, and the cotangent in (8.27) when $\theta = \pi j$, where j is an arbitrary natural number. The general solution in the limit $V_0 \to \infty$ can then be written as $\theta = n\pi/2$, with n being an arbitrary natural number: an even $n = 2j$ produces an odd solution, and an odd $n = 2j+1$ an even solution. Substituting $\theta = ka/2$, we find wavevector values $k_n = n\pi/a$, which correspond to energy eigenvalues

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2ma^2}.$$

We notice that the oscillating parts of the wavefunctions, inside the box,

$$\psi_n(x) \quad = \quad \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), & \text{even } n \\ \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right), & \text{odd } n. \end{cases} ; \tag{8.32}$$

vanish at $x = \pm a/2$. This implies according to Eqs. (8.22) and (8.29) that $B = 0$ for both odd and even cases, and that the wavefunction vanishes outside the box.

We can now find the normalization constant A. To this end, we integrate the square absolute value of the wavefunction over the real axis. We find, for both even and odd solutions,

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x = \int_{-a/2}^{a/2} |\psi(x)|^2 \mathrm{d}x = |A|^2 \frac{a}{2},$$

so the norm is $A = \sqrt{2/a}$.

Solution to Exercise 3.53. Since $V_0 a = W_0$ we can rewrite Eq. (8.25) as

$$\theta_0 = \frac{\sqrt{2mW_0}}{\hbar} \frac{\sqrt{a}}{2}. \tag{8.33}$$

Because a tends to zero and W_0 is a constant, θ_0 also tends to zero. The bold curve in Fig. 8.1 shrinks to a vertical line just next to the ordinates axis. Therefore we have only one, even, energy eigenstate, and we rewrite Eq. (8.31) using the fact that $\tan\theta \approx \theta$ for small θ:

$$\theta = \sqrt{\frac{\theta_0^2}{\theta^2} - 1}. \tag{8.34}$$

or

$$\theta^4 + \theta^2 - \theta_0^2 = 0. \tag{8.35}$$

Therefore

$$\theta^2 = \frac{-1 \pm \sqrt{1 + 4\theta_0^2}}{2}.$$

Because θ_0 is small, we approximate to the *second* order (the reason why we need this will become clear shortly) $\sqrt{1+4\theta_0^2} \approx 1 + 2\theta_0^2 - 2\theta_0^4$. Then the two roots of Eq. (8.35) can be rewritten as

$$\theta^2 \approx \begin{bmatrix} \theta_0^2 - \theta_0^4 \\ -1 - \theta_0^2 + \theta_0^4 \end{bmatrix}. \tag{8.36}$$

Because we are looking for a bound solution, we expect θ to be real, so we choose the first root, $\theta \approx \theta_0$. Since $\theta_0 = \sqrt{2mV_0}a/2\hbar$ and $\theta = \sqrt{2mE}a/2\hbar$, we see that

$$E = \frac{2\hbar^2}{ma^2}\theta^2 = V_0 - \frac{ma^2 V_0^2}{2\hbar^2} = V_0 - \frac{mW_0^2}{2\hbar^2}. \tag{8.37}$$

It is now clear why we needed the second order Taylor expansion. The energy level of the bound state is (almost) at the top of the well. The coefficient κ determining the behavior of the wavefunction

outside the well is given by Eq. (8.21). Had we kept only the first order, we would obtain $E \approx V_0$, and no information about κ. Now we can however calculate

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar} = \frac{W_0 m}{\hbar}. \tag{8.38}$$

As we see, this coefficient is independent of a in the limit $a \to 0$ as long as $V_0 a$ is kept constant.

Let us now find the normalization coefficient. In the limit $a \to 0$, we need to only take into account the part of the wavefunction that is localized outside the well when calculating the norm. Using Eq. (8.28), we have

$$\int_{-\infty}^{+\infty} |\psi(x)^2| = 2B^2 \int_0^{\infty} e^{-2\kappa x} \mathrm{d}x = B^2, \tag{8.39}$$

so $B = \sqrt{\kappa}$.

Solution to Exercise 3.53.

a) Because the potential is an even function of x, we can restrict to even and odd wavefunctions. At $x \neq 0$, the potential is zero. The energy of a bound state must then be negative, so a generic odd solution must be of the form

$$\psi(x) = \begin{cases} -Be^{\kappa x}, & x < 0 \\ Be^{-\kappa x}, & x > 0 \end{cases} \tag{8.40}$$

with $\kappa = \sqrt{2m(-E)}/\hbar$. Unless $B = 0$ (i.e. $\psi(x) \equiv 0$), the odd wavefunction has a discontinuity at $x = 0$, i.e. unphysical.

The even solution is given by

$$\psi(x) = \begin{cases} Be^{\kappa x}, & x < 0 \\ Be^{-\kappa x}, & x > 0 \end{cases}$$

It is continuous at all x, but its derivative has a discontinuity:

$$\Delta\psi'(x)\big|_{x=0} = -2B\kappa.$$

This is not a problem, because the potential has a discontinuity at $x = 0$.

The solution (8.40) is valid for an arbitrary κ at all values of x except $x = 0$. In order to find out for which values of κ the time-independent Schrödinger equation (3.74) is satisfied at $x = 0$, let us integrate this equation over an infinitesimal interval around this point:

$$-\int_{-0}^{+0} \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = \int_{-0}^{+0} [E - V(x)]\psi(x). \tag{8.41}$$

Using the Newton-Leibniz axiom as well as Eq. (C.5), we find

$$-\frac{\hbar^2}{2m} \left(\frac{\mathrm{d}\psi(x)}{dx}\big|_{x=+0} - \frac{\mathrm{d}\psi(x)}{dx}\big|_{x=-0} \right) = W_0 \psi(0). \tag{8.42}$$

Given that, according to Eq. (8.40),

$$\psi(0) = B; \tag{8.43}$$

$$\frac{\mathrm{d}\psi(x)}{dx}\big|_{x=+0} = -B\kappa; \tag{8.44}$$

$$\frac{\mathrm{d}\psi(x)}{dx}\big|_{x=-0} = B\kappa, \tag{8.45}$$

we find from Eq. (8.42) that $\kappa = W_0 m/\hbar^2$ and thus

$$E = \frac{W_0^2 m}{2\hbar^2}.$$

The normalization factor $B = \sqrt{\kappa}$ is found from the condition $\int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x = 1$.

Solution to Exercise 3.54. The particle is initially prepared in the bound state of the original potential (see Ex. 3.53):

$$\psi_0(x) = \sqrt{\kappa_0} \begin{cases} e^{\kappa_0 x}, & x < 0 \\ e^{-\kappa_0 x}, & x > 0 \end{cases}$$

with $\kappa_0 = V_0 m/\hbar^2$. After the sudden change of the potential, the bound state is given by another wavefunction,

$$\psi_1(x) = \sqrt{\kappa_1} \begin{cases} e^{\kappa_1 x}, & x < 0 \\ e^{-\kappa_1 x}, & x > 0 \end{cases}$$

with $\kappa_1 = 2V_0 m/\hbar^2$. The probability that the particle will remain in the bound state of the new potential is given, according to the Second Postulate, by the inner product

$$\begin{aligned} \mathrm{pr} &= |\langle \psi_0 | \, \psi_1 \rangle|^2 \\ &= \left| \int_{-\infty}^{+\infty} \psi_0^*(x) \psi_1(x) \mathrm{d}x \right|^2 \\ &= \kappa_0 \kappa_1 \left| 2 \int_0^{+\infty} e^{-\kappa_0 x} e^{-\kappa_1 x} \mathrm{d}x \right|^2 \\ &= \kappa_0 \kappa_1 \left| \frac{2}{\kappa_0 + \kappa_1} \right|^2 \\ &= \frac{8}{9}. \end{aligned}$$

Solution to Exercise 3.59. The general solution associated with energy eigenvalue E is

$$\psi_{\mathrm{bar}}(E, x) = \begin{cases} A e^{ik_0 x} + B e^{-ik_0 x}, & x < 0 \\ C e^{ik_1 x} + D e^{-ik_1 x}, & x \geq 0 \end{cases}, \tag{8.46}$$

where $k_0 = \sqrt{2mE}/\hbar$, $k_1 = \sqrt{2m(E - V_0)}/\hbar$. The four amplitudes A, B, C, D must be chosen so that the wavefunction and its derivative are continuous at the barrier, i.e.

$$A + B = C + D;$$

$$ik_0(A - B) = ik_1(C - D);$$

One of the parameters provides normalization and can be set to an arbitrary value. This leaves us with three parameters and two equations; therefore, for each energy value, there are two linearly independent solution sets. They can be chosen to correspond to the following physical situations: (1) an initial de Broglie wave approaching from the left

$$B = A\frac{k_0 - k_1}{k_0 + k_1}; \quad C = A\frac{2k_0}{k_0 + k_1}; \quad D = 0, \tag{8.47}$$

and (2) an initial de Broglie wave approaching from the right ($A = 0$)

$$A = 0; \quad B = D\frac{2k_1}{k_0 + k_1}; \quad C = D\frac{k_1 - k_0}{k_0 + k_1}. \tag{8.48}$$

Of course, any linear combination of these solutions is also a solution.

Solution to Exercise 3.62. The initial wavepacket can be written in the wavevector basis as (3.39)

$$|\psi(0)\rangle = \left(\frac{d^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-i\kappa a} e^{-\kappa^2 d^2/2} |k_0+\kappa\rangle \, d\kappa, \tag{8.49}$$

where κ is small compared to k_0 and k_1 (due do the second and fourth assumptions above). To simplify the calculation, we replace the momentum eigenstate $|k\rangle$ in the above equation by an energy eigenstate $|\psi_{\rm bar}(\kappa)\rangle$ (8.46), which we write in the form

$$\langle x| \, \psi_{\rm bar}(\kappa)\rangle = Ae^{i(k_0+\kappa)x}\theta(-x) + Be^{-i(k_0+\kappa)x}\theta(-x) + Ce^{i\sqrt{(k_0+\kappa)^2 - \frac{2mV_0}{\hbar^2}}x}\theta(x). \tag{8.50}$$

In Eq. (8.50) A-wave, with $A = 1/\sqrt{2\pi}$, is identical to $|k_0+\kappa\rangle$, while the B- and C- waves are orthogonal to it. As we shall see below, B- and C-terms do not modify the initial state, but emerge, as separate wavepackets, only after the initial packet reaches the barrier. The amplitude factors B and C are given by Eq. (3.79a). We can neglect the variation of the amplitudes B and C as a function of the small κ. Furthermore, using $k_1^2 = k_0^2 - 2mV_0/\hbar^2$ and neglecting terms that are quadratic with respect to κ, and we can replace in Eq. (8.50)

$$\sqrt{(k_0+\kappa)^2 - \frac{2mV_0}{\hbar^2}} = \sqrt{k_0^2 + 2k_0\kappa - \frac{2mV_0}{\hbar^2}} = k_1\sqrt{1 + 2k_0\kappa/k_1^2} = k_1 + \kappa\frac{k_0}{k_1}.$$

Knowing that the state $|\psi_{\rm bar}(\kappa)\rangle$ is an eigenstate of the Hamiltonian with energy $E_\kappa = \hbar^2(k_0+\kappa)^2/2m \approx (\hbar^2/2m)(k_0^2 + 2k_0\kappa)$, we write the evolution of the state ψ, again neglecting quadratic terms:

$$|\psi(t)\rangle = \left(\frac{d^2}{\pi}\right)^{1/4} e^{-i\hbar k_0^2 t/2m} \int_{-\infty}^{+\infty} e^{-i\kappa(a+\frac{\hbar k_0}{m}t)} e^{-\kappa^2 d^2/2} |\psi_{\rm bar}(\kappa)\rangle \, d\kappa. \tag{8.51}$$

The overall phase factor $e^{-i\hbar k_0^2 t/2m}$ can be neglected.

We shall now calculate the integral (8.51) for each wave in Eq. (8.50) separately.

A-wave. Applying standard Fourier transform rules (C.12), we obtain

$$\begin{aligned}\psi_A(x,t) &= A\theta(-x)e^{ik_0x}\left(\frac{d^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-i\kappa(a+\frac{\hbar k_0}{m}t)} e^{-\kappa^2 d^2/2} e^{i\kappa x} d\kappa \\ &= A\theta(-x)e^{ik_0x}\left(\frac{d^2}{\pi}\right)^{1/4} \frac{\sqrt{2\pi}}{d} e^{-\frac{\left(x-a-\frac{\hbar k_0}{m}t\right)^2}{2d^2}}.\end{aligned}$$

This is a Gaussian wavepaket centered at the point $x = a + \frac{\hbar k_0}{m}t$ and propagating with the speed $\hbar k_0/m$ in the positive direction. When the wavepacket reaches the barrier (i.e. at $t_{\rm bar} = \frac{-am}{\hbar k_0}$), it disappears due to the factor $\theta(-x)$. Before this happens, the total probability associated with this wavepacket is $\mathrm{pr_A} = \int_{-\infty}^{+\infty} |\psi_A(x,t)|^2 dx = 2\pi A^2 = 1$.

B-wave is treated similarly, except that the integral is the inverse Fourier transform. We obtain

$$\psi_B(x,t) = B\theta(-x)e^{-ik_0x}\left(\frac{d^2}{\pi}\right)^{1/4} \frac{\sqrt{2\pi}}{d} e^{-\frac{\left(x+a+\frac{\hbar k_0}{m}t\right)^2}{2d^2}}.$$

This wavepaket is a mirror image of the previous one. At $t = 0$, it is located at $x = -a$ but is "invisible" due to the factor $\theta(-x)$. It propagates in the negative direction. Once it reaches the barrier (simultaneously with the A-packet), it becomes "visible". This wavepacket is associated with the reflection of the particle from the barrier. The total probability associated with this wavepacket is $\mathrm{pr_B} = 2\pi B^2$.

C-wave.

$$\psi_C(x,t) = C\theta(x)e^{ik_1x}\left(\frac{d^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-i\kappa(a+\frac{\hbar k_0}{m}t)} e^{-\kappa^2 d^2/2} e^{i\kappa\frac{k_0}{k_1}x} d\kappa$$

$$= C\theta(x)e^{ik_1x}\left(\frac{d^2}{\pi}\right)^{1/4}\frac{\sqrt{2\pi}}{d}e^{-\frac{\left(\frac{k_0}{k_1}x-a-\frac{\hbar k_0}{m}t\right)^2}{2d^2}}.$$

This packet is narrower than the other two by a factor k_0/k_1. It begins to exist at $t = t_{\rm bar}$ and propagates in the positive direction at a speed $\hbar k_1/m$. This wavepacket is associated with the particle transmitted through the barrier and has the probability $\mathrm{pr_C} = 2\pi C^2 k_1/k_0$. A direct calculation shows that $\mathrm{pr_B} + \mathrm{pr_C} = 1$.

Solution to Exercise 3.63. Proceeding similarly to Ex. 3.59 we find that the solution is a linear combination of six wavefunctions as shown in Fig. 3.7 and is thus a function of six parameters. For each of the two interfaces, there are two continuity conditions (for the wavefunction and its derivative):

$$A + B = C + D;$$

$$ik_0(A - B) = k_1(C - D);$$

$$Ce^{k_1L} + De^{-k_1L} = E + F;$$

$$k_1(Ce^{k_1L} - De^{-k_1L}) = ik_0(E - F),$$

where $k_0 = \sqrt{2mE}/\hbar$, $k_1 = \sqrt{2m(V_0 - E)}/\hbar$. Again, each energy value is twice degenerate: the linearly independent solutions correspond to the matter waves approaching from the left ($F = 0$) and from the right ($A = 0$). We are interested in the first option and solve the above equations assuming an arbitrary E and working our way to the left. We then find the relation between the incident, transmitted and reflected amplitudes:

$$A = E\left[\cosh k_1L + \frac{i}{2}\left(\frac{k_1}{k_0} - \frac{k_0}{k_1}\right)\sinh(k_1L)\right]; \tag{8.52}$$

$$B = E\left[-\frac{i}{2}\left(\frac{k_1}{k_0} + \frac{k_0}{k_1}\right)\sinh(k_1L)\right]. \tag{8.53}$$

The transmission and reflection coefficients are then given by Eqs. (3.81).

Solution to Exercise 3.67. We rescale the position and momentum operators according to $\hat{X} = A\hat{x}$, $\hat{P} = B\hat{p}$. We then have, for the counterpart classical observables, $P/X = (B/A)(p/x)$. Because $p_{max}/x_{max} = m\omega$, we find $P_{max}/X_{max} = (B/A)m\omega$. The requirement that $P_{max} = X_{max}$ implies that

$$\frac{A}{B} = m\omega.$$

On the other hand, the commutator of the rescaled observables satisfies $[\hat{X}, \hat{P}] = AB[\hat{x}, \hat{p}] = i\hbar AB$. Since we need this commutator to equal i, we obtain the second equation:

$$AB = \frac{1}{\hbar}.$$

Solving these two equations for A and B, we find the desired result.

Solution to Exercise 3.68. Since $\hbar$ has the same dimension as the product of the position and momentum, i.e. kg·m^2/s, the dimension of $\sqrt{m\omega/\hbar}$ is m^{-1} (i.e. the same as x^{-1}) and that of $\sqrt{m\omega\hbar}$ is kg·m/s (i.e. the same as p).

Solution to Exercise 3.69. As discussed in Sec. 3.2, rescaling a continuous variable affects the normalization of its eigenstate: if $\hat{X} = \hat{x}\sqrt{m\omega/\hbar}$, then $|X\rangle = (\hbar/m\omega)^{1/4}|x\rangle$. Similarly, the renormalized eigenstatewe of the rescaled momentum $\hat{P} = \hat{p}/\sqrt{m\hbar\omega}$ is $|P\rangle = (m\hbar\omega)^{1/4}|p\rangle$. Now for the de Broglie wave we have

$$\langle X|\,P\rangle \;=\; \left(\frac{\hbar}{m\omega}\right)^{1/4}(m\hbar\omega)^{1/4}\langle x|\,p\rangle = \sqrt{\hbar}\frac{1}{\sqrt{2\pi\hbar}}e^{ixp/\hbar} = \frac{1}{\sqrt{2\pi}}e^{iXP}.$$

Using this result, we argue similarly to Ex. 3.13 to obtain the conversion expressions for wavefunctions in the rescaled observable basis:

$$\begin{aligned}\psi(X) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{\psi}(P)e^{iPX}\mathrm{d}P;\\ \tilde{\psi}(P) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi(X)e^{-iPX}\mathrm{d}X.\end{aligned}$$

Solution to Exercise 3.70. Using the relations found in Ex. 3.69, we find

$$\begin{aligned}\left\langle X\right|\hat{P}\left|\,\psi\right\rangle &= \left(\frac{\hbar}{m\omega}\right)^{1/4}\frac{1}{\sqrt{m\hbar\omega}}\left\langle x\right|\hat{p}\left|\,\psi\right\rangle\\ &\overset{(3.41)}{=} \left(\frac{\hbar}{m\omega}\right)^{1/4}\frac{1}{\sqrt{m\hbar\omega}}(-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}\left\langle x\right|\,\psi\rangle\\ &= \frac{1}{\sqrt{m\hbar\omega}}(-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}\left\langle X\right|\,\psi\rangle\\ &= \frac{1}{\sqrt{m\hbar\omega}}(-i\hbar)\sqrt{\frac{m\omega}{\hbar}}\frac{\mathrm{d}}{\mathrm{d}X}\left\langle X\right|\,\psi\rangle\\ &= -i\frac{\mathrm{d}}{\mathrm{d}X}\left\langle X\right|\,\psi\rangle.\end{aligned}$$

The expression for the position operator in the momentum basis is obtained similarly.

Solution to Exercise 3.71.

$$H = \frac{\hat{p}^2}{2m}+\frac{m\omega^2\hat{x}^2}{2} = m\omega\hbar\frac{\hat{P}^2}{2m}+\frac{\hbar}{m\omega}\frac{m\omega^2\hat{X}^2}{2} = \frac{1}{2}\hbar\omega\left(\hat{X}^2+\hat{P}^2\right). \tag{8.54}$$

Solution to Exercise 3.72.

a) Because the position and momentum operators are Hermitian, $\hat{X}^\dagger = \hat{X}$ and $(i\hat{P})^\dagger = -i\hat{P}$. Therefore, $\hat{a}^\dagger = (\hat{X}+i\hat{P})^\dagger/\sqrt{2} = (\hat{X}-i\hat{P})/\sqrt{2}$.

b)

$$[\hat{a},\hat{a}^\dagger] = \frac{1}{2}[\hat{X}+i\hat{P},\hat{X}-i\hat{P}] = \frac{1}{2}\big([\hat{X},\hat{X}]-i[\hat{X},\hat{P}]+i[\hat{P},\hat{X}]+[\hat{P},\hat{P}]\big) = 1.$$

c) The position and momentum operators are expressed through $\hat{a}$ and $\hat{a}^\dagger$ by solving Eqs. (3.86) and (3.87).

d)

$$\begin{aligned}\hat{H} &= \frac{1}{2}\hbar\omega(\hat{X}^2+\hat{P}^2)\\ &= \frac{1}{4}\hbar\omega\left[(\hat{a}+\hat{a}^\dagger)^2+\frac{1}{i^2}(\hat{a}-\hat{a}^\dagger)^2\right]\\ &= \frac{1}{4}\hbar\omega\left[(\hat{a}^2+(\hat{a}^\dagger)^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a})+\frac{1}{i^2}(\hat{a}^2+(\hat{a}^\dagger)^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a})\right]\\ &= \frac{1}{4}\hbar\omega[2\hat{a}\hat{a}^\dagger+2\hat{a}^\dagger\hat{a}]\\ &\overset{(3.88)}{=} \frac{1}{4}\hbar\omega[2\hat{a}^\dagger\hat{a}+2+2\hat{a}^\dagger\hat{a}]\\ &= \hbar\omega\left[\hat{a}^\dagger\hat{a}+\frac{1}{2}\right].\end{aligned}$$

e)

$$[\hat{a}, \hat{a}^\dagger \hat{a}] \overset{(1.52)}{=} \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}, \hat{a}^\dagger]\hat{a} = \hat{a};$$

$$[\hat{a}^\dagger, \hat{a}^\dagger \hat{a}] \overset{(1.52)}{=} \hat{a}^\dagger [\hat{a}^\dagger, \hat{a}] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} = -\hat{a}^\dagger.$$

Solution to Exercise 3.73.

a) In order to verify if the state $\hat{a}\,|n\rangle$ is an eigenstate of the photon number operator $\hat{a}^\dagger\hat{a}$, let us subject this state to the action of this operator and employ the result (3.91), rewritten in the form $\hat{a}^\dagger\hat{a}\hat{a} = \hat{a}\hat{a}^\dagger\hat{a} - \hat{a}$:

$$\hat{a}^\dagger\hat{a}\hat{a}\,|n\rangle = (\hat{a}\hat{a}^\dagger\hat{a} - \hat{a})\,|n\rangle = (\hat{a}n - \hat{a})\,|n\rangle = (n-1)\hat{a}\,|n\rangle\,,$$

as was required.

b) Similarly, from Eq. (3.91) we find $\hat{a}^\dagger\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a}^\dagger\hat{a} + \hat{a}^\dagger$ and thus

$$\hat{a}^\dagger\hat{a}\hat{a}^\dagger\,|n\rangle = (\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \hat{a}^\dagger)\,|n\rangle = (\hat{a}^\dagger n + \hat{a}^\dagger)\,|n\rangle = (n+1)\hat{a}^\dagger\,|n\rangle\,.$$

Solution to Exercise 3.74.

a) Let $|\psi\rangle = \hat{a}\,|n\rangle$. From the previous exercise, we know that $|\psi\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue $n-1$, i.e. $|\psi\rangle = A\,|n-1\rangle$, where A is some constant. We need to find A. To this end, we notice that $\langle\psi| = \langle n|\,\hat{a}^\dagger$ and calculate

$$\langle\psi|\,\psi\rangle = \big\langle n\big|\,\hat{a}^\dagger\hat{a}\big|\,n\big\rangle = n.$$

But on the other hand,

$$\langle\psi|\,\psi\rangle = |A|^2\,\langle n-1|\,n-1\rangle = |A|^2,$$

where in the last equality we have used the fact that the eigenstates of the number operator are normalized. From the last two equations, we find $|A| = \sqrt{n}$.

The phase of A is arbitrary. By convention, it is chosen equal to zero, so A is real and positive: $A = \sqrt{n}$.

b) Similarly, if $|\phi\rangle = \hat{a}^\dagger\,|n\rangle = B\,|n+1\rangle$, then, on one hand,

$$\langle\phi|\,\phi\rangle = \big\langle n\big|\,\hat{a}\hat{a}^\dagger\big|\,n\big\rangle = \big\langle n\big|\,\hat{a}^\dagger\hat{a} + 1\big|\,n\big\rangle = n+1,$$

and on the other hand

$$\langle\phi|\,\phi\rangle = |B|^2\,\langle n+1|\,n+1\rangle = |B|^2.$$

Therefore (invoking a similar convention), $B = \sqrt{n+1}$.

Solution to Exercise 3.75.

$$|n\rangle \overset{(3.93b)}{=} \frac{\hat{a}^\dagger}{\sqrt{n}}\,|n-1\rangle = \frac{\hat{a}^\dagger}{\sqrt{n}}\frac{\hat{a}^\dagger}{\sqrt{n-1}}\,|n-2\rangle = \ldots = \frac{\big(\hat{a}^\dagger\big)^n}{\sqrt{n!}}\,|0\rangle\,.$$

Solution to Exercise 3.76. The vacuum state obeys the equation $\hat{a}\,|0\rangle = 0$, or

$$(\hat{X} + i\hat{P})\,|0\rangle = 0. \tag{8.55}$$

In order to find the wavefunction in the position basis, we use Eq. (3.84) to write the momentum operator in this basis. Equation (8.55) then becomes

$$\left(X + \frac{\mathrm{d}}{\mathrm{d}X}\right)\psi(X) = 0.$$

This is a first order ordinary differential equation whose solution is

$$\psi(x) = Ae^{-X^2/2},$$

where A is the normalization constant, calculated in the usual manner:

$$\langle\psi|\ \psi\rangle = \int_{-\infty}^{+\infty} |\psi(X)|^2 \mathrm{d}x = |A|^2 \int_{-\infty}^{+\infty} e^{-X^2} \mathrm{d}x = |A|^2\sqrt{\pi}.$$

Requiring the norm of $|\psi\rangle$ to equal 1, we find $A = \pi^{-1/4}$.

The wavefunction in the momentum basis is calculated similarly.

Solution to Exercise 3.77.

a) The single-photon Fock state is obtained from the vacuum state by applying a single creation operator. Using Eq. (3.84), we express the creation operator in the position basis as

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) \leftrightarrow \frac{1}{\sqrt{2}}\left(X - \frac{\mathrm{d}}{\mathrm{d}X}\right)$$

and thus the wavefunction of the state $|1\rangle = \hat{a}^\dagger |0\rangle$ is

$$\psi_1(X) = \frac{1}{\sqrt{2}\pi^{1/4}}\left(X - \frac{\mathrm{d}}{\mathrm{d}X}\right) e^{-X^2/2} = \frac{\sqrt{2}}{\pi^{1/4}} X e^{-X^2/2}.$$

The two-photon Fock state is obtained by applying the creation operator to the single-photon state:

$$|2\rangle \overset{(3.93\mathrm{b})}{=} \frac{\hat{a}^\dagger}{\sqrt{2}} |1\rangle .$$

In the position basis,

$$\psi_2(X) = \frac{1}{2}\left(X - \frac{\mathrm{d}}{\mathrm{d}X}\right)\psi_1(x) = \frac{1}{\sqrt{2}\pi^{1/4}}\left(X - \frac{\mathrm{d}}{\mathrm{d}X}\right) X e^{-X^2/2} = \frac{1}{\sqrt{2}\pi^{1/4}}(2X^2 - 1)e^{-X^2/2}.$$

b) We now show by induction that Eq. (3.97) describes the wavefunction of the Fock state $|n\rangle$. First, from the definition of the Hermite polynomial,

$$H_n(X) = (-1)^n e^{X^2} \frac{\mathrm{d}^n}{\mathrm{d}X^n} e^{-X^2},$$

we find that $H_0(X) = 1$ and thus the wavefunction of the vacuum state obtained from Eq. (3.97) is $\psi_0(X) = \pi^{-1/4} e^{-X^2/2}$, which is consistent with Eq. (3.96). Second, assuming that if Eq. (3.97) is valid for a specific Fock state $|n\rangle$, we need to prove it to be also valid for the next Fock state $|n+1\rangle = \hat{a}^\dagger |n\rangle / \sqrt{n+1}$. We apply the creation operator in the position basis:

$$\begin{aligned}
|n+1\rangle &= \frac{\hat{a}^\dagger}{\sqrt{n+1}} |n\rangle \\
&\leftrightarrow \frac{X - \mathrm{d}/\mathrm{d}X}{\sqrt{2}\sqrt{n+1}} \frac{H_n(X)}{\pi^{1/4}\sqrt{2^n n!}} e^{-X^2/2} \\
&= \frac{1}{\pi^{1/4}\sqrt{2^{n+1}(n+1)!}} \left[X H_n(X) e^{-X^2/2} - \frac{\mathrm{d}H_n(X)}{\mathrm{d}X} e^{-X^2/2} - H_n(X) \frac{\mathrm{d}e^{-X^2/2}}{\mathrm{d}X} \right] \\
&= \frac{1}{\pi^{1/4}\sqrt{2^{n+1}(n+1)!}} \left[\left(2X H_n(X) - \frac{\mathrm{d}H_n(X)}{\mathrm{d}X} \right) e^{-X^2/2} \right] \\
&= \frac{1}{\pi^{1/4}\sqrt{2^{n+1}(n+1)!}} H_{n+1}(X) e^{-X^2/2},
\end{aligned}$$

which, according to Eq. (3.96), is the wavefunction of the state $|n+1\rangle$. In the final step in the above transformation, we have applied the known recursion relation for the Hermite polynomials,

$$H_{n+1}(X) = 2X\, H_n(X) - \frac{\mathrm{d}H_n(X)}{\mathrm{d}X} H_n(X).$$

Solution to Exercise 3.78. For an arbitrary Fock state $|n\rangle$, we have

$$\begin{aligned} \left\langle n\right| \hat{X} \left| n\right\rangle &= \frac{1}{\sqrt{2}} \langle n| \left(\hat{a} + \hat{a}^\dagger\right) |n\rangle \\ &= \frac{1}{\sqrt{2}} \langle n| \left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right) \\ &= 0. \end{aligned} \tag{8.56}$$

Similarly,

$$\left\langle n\right| \hat{P} \left| n\right\rangle = 0.$$

For the uncertainties, we have

$$\begin{aligned} \langle \Delta X^2 \rangle &= \left\langle n\right| \hat{X}^2 \left| n\right\rangle \\ &= \frac{1}{2} \langle n| \left(\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\right) |n\rangle \\ &= \frac{1}{2} \langle n| \left[\sqrt{n(n-1)}\,|n-2\rangle + \sqrt{n+1}^2\,|n\rangle + \sqrt{n}^2\,|n\rangle + \sqrt{(n+1(n+2)}\,|n+2\rangle\right] \\ &= \frac{1}{2}(2n+1). \end{aligned} \tag{8.57}$$

The same answer holds for the momentum uncertainty:

$$\langle \Delta P^2 \rangle = \frac{1}{2}(2n+1).$$

Solution to Exercise 3.79. Both $|0\rangle$ and $|1\rangle$ are energy eigenstates with the eigenvalues being, respectively, $\hbar\omega/2$ and $3\hbar\omega/2$. The evolution of the superposition of these states is then given by

$$|\psi(t)\rangle = e^{-i\omega t/2}\,|0\rangle + e^{-3i\omega t/2}\,|1\rangle\,.$$

The expectation value of the position observable is then given by

$$\langle X\rangle \overset{(3.89)}{=} \frac{1}{\sqrt{2}} \langle\psi(t)| \left(\hat{a} + \hat{a}^\dagger\right) |\psi(t)\rangle = \frac{1}{\sqrt{2}} \big(\langle 0| + e^{i\omega t} \langle 1| \big) \big(\hat{a} + \hat{a}^\dagger\big) \big(|0\rangle + e^{-i\omega t}\,|1\rangle \big).$$

The only nonvanishing matrix elements in the expression above are $\langle 0|\, \hat{a}\, |1\rangle = \left\langle 1\right| \hat{a}^\dagger \left| 0\right\rangle = 1$. We then conclude that

$$\langle X\rangle = \frac{1}{\sqrt{2}} \big(e^{i\omega t} + e^{-i\omega t}\big) = \sqrt{2}\cos\omega t.$$

Similarly, for the momentum observable we find

$$\langle X\rangle = \frac{1}{\sqrt{2}i} \big(\langle 0| + e^{i\omega t} \langle 1| \big) \big(\hat{a} - \hat{a}^\dagger\big) \big(|0\rangle + e^{-i\omega t}\,|1\rangle \big) = \frac{1}{\sqrt{2}i} \big(- e^{i\omega t} + e^{-i\omega t}\big) = -\sqrt{2}\sin\omega t.$$

The trajectory in the phase space is a clockwise circle with the center in the origin and a radius of $\sqrt{2}$.

Solution to Exercise 3.80. We will perform the calculation in the position basis. Acting similarly to Ex. 3.76, we rewrite Eq. (3.99) as

$$\frac{1}{\sqrt{2}} \left(X + \frac{\mathrm{d}}{\mathrm{d}X} \right) \psi(X) = (\mathrm{Re}\,\alpha + i\mathrm{Im}\,\alpha)\psi(X).$$

Substituting Eq. (3.100), we find

$$\begin{aligned}\frac{1}{\sqrt{2}}\left(X+\frac{\mathrm{d}}{\mathrm{d}X}\right)\psi_\alpha(X) &= \frac{1}{\sqrt{2}\pi^{1/4}}e^{i\frac{P_\alpha X_\alpha}{2}}\left(X+\frac{\mathrm{d}}{\mathrm{d}X}\right)e^{iP_\alpha X}e^{-\frac{(X-X_\alpha)^2}{2}} \\ &= \frac{1}{\sqrt{2}\pi^{1/4}}e^{i\frac{P_\alpha X_\alpha}{2}}\left[X+iP_\alpha-(X-X_\alpha)\right]e^{iP_\alpha X}e^{-\frac{(X-X_\alpha)^2}{2}} \\ &= \frac{1}{\sqrt{2}}\left(X_\alpha+iP_\alpha\right)\psi_\alpha(X), \end{aligned} \tag{8.58}$$

so Eq. (3.99) holds provided that $X_\alpha = \sqrt{2}\mathrm{Re}\,\alpha$ and $P_\alpha = \sqrt{2}\mathrm{Im}\,\alpha$.

The wavefunction (3.101) in the momentum basis is obtained from that in the position basis by means of Fourier transform, similarly to Ex. 3.16.

Solution to Exercise 3.81. The right-hand side of Eq. (3.103) consists of sequential application of the position and momentum displacement operators, as well as an overall phase factor, to the vacuum state Let us first just apply the position displacement:

$$|\psi'\rangle = e^{-iX_\alpha\hat{P}}|0\rangle\,.$$

Working in the position basis, we obtain the wavefunction

$$\psi'(x) = \psi_0(X-X_\alpha) \overset{(3.96)}{=} \frac{1}{\pi^{1/4}}e^{-(X-X_\alpha)^2/2}.$$

Now if we apply the momentum displacement,

$$|\psi''\rangle = e^{iP_\alpha\hat{X}}|\psi'\rangle\,,$$

we have

$$\psi''(x) = \frac{1}{\pi^{1/4}}e^{iP_\alpha X}e^{-(X-X_\alpha)^2/2}.$$

Upon multiplying by the phase factor $e^{-iP_\alpha X_\alpha/2}$, we obtain the wavefunction that is identical to that of the coherent state (3.100).

The proof of the second equality in Eq. (3.103) is left for the reader as an independent exercise.

Solution to Exercise 3.82. The right-hand side of Eq. (3.103) consists of sequential applications of the position displacement by X_α and momentum displacement by P_α. As we know from Ex. 3.21 and 3.22, each of these operators adds the displacement value to the mean value of the position or momentum, respectively, while leaving the expectation value of the other canonical observable as well as the uncertainties of both observables. Based on this fact, and using Eq. (3.102), we obtain the desired result.

Solution to Exercise 3.83. Let us apply the Baker-Hausdorff-Campbell formula (1.70) to Eq. (3.103). We have $\hat{A} = iP_\alpha\hat{X}$ and $\hat{B} = -iX_\alpha\hat{P}$. Then the commutator

$$[\hat{A},\hat{B}] = P_\alpha X_\alpha[\hat{X},\hat{P}] = iP_\alpha X_\alpha$$

is a number (which we denote as c), so the Baker-Hausdorff-Campbell formula is applicable. In our notation, the right-hand side of Eq. (3.103) is identical to the right-hand side of Eq. (1.70), so we can write

$$|\alpha\rangle = e^{\hat{A}+\hat{B}}|0\rangle = e^{iP_\alpha\hat{X}-iX_\alpha\hat{P}}|0\rangle\,. \tag{8.59}$$

Let us now transform this result as follows.

$$\begin{aligned}|\alpha\rangle &= \exp(iP_\alpha\hat{X}-iX_\alpha\hat{P})|0\rangle \\ &= \exp\left(\frac{\hat{X}_\alpha+i\hat{P}_\alpha}{\sqrt{2}}\frac{\hat{X}-i\hat{P}}{\sqrt{2}}-\frac{\hat{X}_\alpha-i\hat{P}_\alpha}{\sqrt{2}}\frac{\hat{X}+i\hat{P}}{\sqrt{2}}\right)|0\rangle \\ &= \exp\left(\alpha\hat{a}^\dagger-\alpha^*\hat{a}\right)|0\rangle\,. \end{aligned} \tag{8.60}$$

Now since the commutator

$$[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}] = -|\alpha^2|[\hat{a}^\dagger, \hat{a}] = |\alpha^2|,$$

is also a number, we can use the Baker-Hausdorff-Campbell formula again and obtain Eq. (3.106).

In order to simplify this equation, let us decompose

$$e^{-\alpha^*\hat{a}}\,|0\rangle = \sum_{n=0}^{\infty} \frac{(-\alpha^*)^n}{n!}\hat{a}^n\,|0\rangle = |0\rangle\,.$$

The last equality above holds because, since $\hat{a}$ is the annihilation operator, all the terms in the sum vanish except for $n = 0$.

Substituting this result into Eq. (3.106), we obtain Eq. (3.107).

Solution to Exercise 3.84. Decomposing Eq. (3.107) into the Taylor series, we find

$$\begin{aligned} |\alpha\rangle &= e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}\,|0\rangle \\ &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(\hat{a}^\dagger)^n\,|0\rangle \\ &\overset{(3.95)}{=} e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \end{aligned} \tag{8.61}$$

The same result can also be partially obtained using a more intuitive argument, not involving displacement operators. Let us assume some decomposition of the coherent state into the number basis,

$$|\alpha\rangle = \sum_{n=0}^{\infty}\alpha_n\,|n\rangle\,, \tag{8.62}$$

and apply the definition (3.99) of the coherent state to this decomposition. For the left-hand side of Eq. (3.99), we have in accordance with Eq. (3.93a),

$$\begin{aligned} \hat{a}\,|\alpha\rangle &= \sum_{n=0}^{\infty}\alpha_n\hat{a}\,|n\rangle \\ &= \sum_{n=1}^{\infty}\alpha_n\sqrt{n}\,|n-1\rangle \\ &\overset{n'=n-1}{=} \sum_{n'=0}^{\infty}\alpha_{n'+1}\sqrt{n'+1}\,|n'\rangle\,. \end{aligned} \tag{8.63}$$

At the same time, the right-hand side of (3.99) can be written as

$$\alpha\,|\alpha\rangle = \sum_{n'=0}^{\infty}\alpha\alpha_{n'}\,|n'\rangle\,. \tag{8.64}$$

Equalizing both sides, we find a recursive relation

$$\alpha_{n'+1} = \frac{\alpha\alpha_{n'}}{\sqrt{n'+1}}, \tag{8.65}$$

so

$$\begin{aligned} \alpha_1 &= \alpha\alpha_0; \\ \alpha_2 &= \frac{\alpha\alpha_1}{\sqrt{2}} = \frac{\alpha^2\alpha_0}{\sqrt{2}}; \\ \alpha_3 &= \frac{\alpha\alpha_2}{\sqrt{3}} = \frac{\alpha^3\alpha_0}{\sqrt{6}}; \\ &\ldots, \end{aligned} \tag{8.66}$$

or in general

$$\alpha_n = \frac{\alpha^n \alpha_0}{\sqrt{n!}}. \tag{8.67}$$

It remains to find such a value of α_0 that state (8.62) is normalized to one. We find

$$\langle \alpha |\, \alpha \rangle = \sum_{n=0}^{\infty} |\alpha_n|^2 = |\alpha_0|^2 \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!}. \tag{8.68}$$

If we look carefully at the sum in the above expression, we will find it to be the Taylor decomposition of $e^{|\alpha|^2}$, so we have $\langle \alpha |\, \alpha \rangle = |\alpha_0|^2 e^{|\alpha|^2}$. Setting $\langle \alpha |\, \alpha \rangle = 1$, we find

$$|\alpha_0|^2 = e^{-|\alpha|^2} \tag{8.69}$$

or

$$\alpha_0 = e^{i\varphi_\alpha} e^{-|\alpha|^2/2}. \tag{8.70}$$

The quantum phase factor $e^{i\varphi_\alpha}$ is a matter of convention, but it must be consistent with the convention chosen for the phase of the coherent state wavefunction (3.100). This phase cannot be determined based on these elementary considerations. However, a rigorous derivation shown in the beginning of this solution demonstrates that $\varphi_\alpha = 0$ for all α.

Combining Eqs. (8.67) and (8.70) we obtain

$$\alpha_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}. \tag{8.71}$$

Solution to Exercise 3.85. For the mean number of excitation quanta, we write:

$$\begin{aligned}
\langle n \rangle &= \sum_{n=0}^{\infty} n \mathrm{pr}_n \\
&= e^{-|\alpha|^2} \sum_{n=0}^{\infty} n \frac{|\alpha|^{2n}}{n!} \\
&= e^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{|\alpha|^{2n}}{(n-1)!} \\
&\overset{n'=n-1}{=} |\alpha|^2 e^{-|\alpha|^2} \sum_{n'=0}^{\infty} \frac{|\alpha|^{2n'}}{n'!}.
\end{aligned} \tag{8.72}$$

As we know from calculus, the sum equals $e^{|\alpha|^2}$. Accordingly, $\langle n \rangle = |\alpha|^2$.

For the mean square number of excitation quanta, the calculation is similar, but somewhat more complicated:

$$\begin{aligned}
\langle n^2 \rangle &= e^{-|\alpha|^2} \sum_{n=0}^{\infty} n^2 \frac{|\alpha|^{2n}}{n!} \\
&= e^{-|\alpha|^2} \sum_{n=1}^{\infty} n \frac{|\alpha|^{2n}}{(n-1)!} \\
&\overset{n'=n-1}{=} |\alpha|^2 e^{-|\alpha|^2} \sum_{n'=0}^{\infty} (n'+1) \frac{|\alpha|^{2n'}}{n'!} \\
&= |\alpha|^2 e^{-|\alpha|^2} \left[\sum_{n'=0}^{\infty} n' \frac{|\alpha|^{2n'}}{n'!} + \sum_{n'=0}^{\infty} \frac{|\alpha|^{2n'}}{n'!} \right] \\
&= |\alpha|^2 \left[\sum_{n'=0}^{\infty} n' \mathrm{pr}_{n'} + \sum_{n'=0}^{\infty} \mathrm{pr}_{n'} \right] \\
&\overset{(8.72)}{=} |\alpha|^2 \left[|\alpha|^2 + 1 \right].
\end{aligned} \tag{8.73}$$

Hence the variance of n is $\langle \Delta n^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2 = |\alpha|^2$.

Solution to Exercise 3.86.

a) The energy of each pulse is $E_p = P/f = 10^{-8}$ J.

b) The energy of each photon is

$$\hbar\omega = \hbar \frac{2\pi c}{\lambda} = 2.5 \times 10^{-19} \text{ J}.$$

The mean number of photons per pulse is $\langle n \rangle = E_p/\hbar\omega = 4 \times 10^{10}$.

c) The variance in the number of photons per pulse equals $\langle \Delta n^2 \rangle = \langle n \rangle$. Hence the uncertainty in this number equals $\sqrt{\langle \Delta n^2 \rangle} = 2 \times 10^5$. The relative uncertainty is the inverse of this number: $\sqrt{\langle \Delta n^2 \rangle}/\langle n \rangle = 5 \times 10^{-6}$.

If we reduce the laser power by a factor of 10^6, the mean number of photons per pulse will reduce by the same factor. On the other hand, the uncertainty in that number will only reduce by a factor of 10^3. Therefore, the relative uncertainty will increase by a factor of 10^3 and become 0.005.

Solution to Exercise 3.89. Given that the coherent state is decomposed into the Fock basis according to Eq. (3.108) and that each Fock state is an eigenstate of the Hamiltonian with eigenvalue $\hbar\omega(n + 1/2)$, we find

$$\begin{aligned} e^{-i\hat{H}t/\hbar} |\alpha\rangle &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\hat{H}t/\hbar} |n\rangle \\ &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega(n+1/2)t} |n\rangle \\ &= e^{-i\omega t/2} e^{-|\alpha|^2/2} \sum_n \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle \\ &= e^{-i\omega t/2} \left| e^{-i\omega t} \alpha \right\rangle . \end{aligned} \tag{8.74}$$